Grade 7/8 Math Circles
Oct 3/4/5/6, 2022
Recursive Sequences

Introduction

In mathematics, we typically use the word **pattern** to describe a rule or property that relates different objects (numbers, data, shapes, etc.).

Warm-Up

Identify the following patterns:

i) 1, 3, 5, 7, 9, ...

ii) 1, 2, 4, 8, 16, ...

iii) 1, 3, 7, 15, 31, ...

iv) Blue, Euclid, Dilemma, Amicable, Elf, Floor, Roof, Fort, ...

Warm-Up Solutions

i) Every number is obtained by adding 2 to the previous number.

ii) Every number is obtained by multiplying the previous number by 2.

iii) Every number is obtained by multiplying the previous number by 2 and adding 1. There are other possible solutions: eg. the difference between terms increases by a factor of 2.

iv) The last letter of every word is the first letter of the next word. Further, the second last letter of every word is the second letter of the next word. Eg. from 'Blue' , take 'ue', flip them to get 'eu' and then 'Euclid' starts with 'eu'.

Sequences

A **sequence** is an ordered list of numbers. For example,

$$1,\ -2,\ 3,\ -4,\ 5,\ -6,\ 7$$

is a finite sequence of length 7. We can also have infinite sequences, such as

$$1,\ -2,\ 3,\ -4,\ 5,\ -6,\ 7,\ -8,\ 9,\ -10,\ 11,\ ...$$

The ... at the end of the sequence indicates that it will continue indefinitely, but we cannot assume that the rest of the sequence follows the same pattern. For example, 1, 2, 3, 4, 5, ... might continue as

$$1,\ 2,\ 3,\ 4,\ 5,\ 8,\ 7,\ 16,\ 9,\ 32,\ ...$$

do you notice a new pattern?

The n^{th} number in a sequence is called the n^{th} **term** of that sequence. In the above sequence, the 1^{st} term is 1, the 2^{nd} term is 2, and the 6^{th} term is 8.

To describe sequences, we use a variable (usually a letter, eg. t for term) and a subscript to indicate the term number. For example,

$$\{t_n\} = t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ ...$$

is a sequence of unknown values. $\{t_n\}$ is used to represent the entire sequence (rather than writing out t_1, t_2, ... every time).

Exercise 1

Find the 6^{th} term in each of the sequences.

i) the sequence which starts with 1 and increases by 1 every term

ii) $t_n = \frac{1}{n}$ for $n \geq 1$

iii) $\{a_n\} = \{n + 3\}$

iv) 1, 3, 5, 7, 9, ...

Exercise 1 Solution

i) The first 6 terms of this sequence are 1, 2, 3, 4, 5, 6 so the 6^{th} term is 6.

ii) The 6th term is $t_6 = \frac{1}{6}$.

iii) This is the same as writing $a_n = n + 3$ for $n \geq 1$. The 6^{th} term is $a_6 = 6 + 3 = 9$.

iv) This is a trick question! We cannot find the 6^{th} term of this sequence since we have not defined anything beyond the first five terms.

When working with sequences, the letter n represents the term number. Thus, if we have a term t_n in a sequence, t_{n-1} is the previous term, and t_{n+1} is the next term. Similarly, we can use an expression in terms of n to denote a different term number: eg. t_{2n} is the $2n^{th}$ term in the sequence.

Exercise 2

Let $\{t_n\} = \{3n + 5\}$.

a) Find the first five terms of this sequence.

b) If $n = 4$, find t_n, t_{2n} and t_{3n+5}.

c) True or False: $t_{n+3} = t_n + 9$ for all $n \geq 1$.

Exercise 2 Solution

a) Using the formula $t_n = 3n + 5$, we find that the first five terms are 8, 11, 14, 17, 20.

b) $n = 4$, $2n = 2(4) = 8$ and $3n + 5 = 3(4) + 5 = 17$ so

$$t_n = t_4 = 3 \times 4 + 5 = 17$$
$$t_{2n} = t_8 = 3 \times 8 + 5 = 29$$
$$t_{3n+5} = t_{17} = 3 \times 17 + 5 = 56$$

c) True: for all n,

$$t_{n+3} = 3(n + 3) + 5 = 3n + 9 + 5 = (3n + 5) + 9 = t_n + 9$$

Different Types of Sequences

Sometimes, we define sequences as being an ordered list of objects (rather than numbers). This allows us to create sequences of shapes, sequences of words, even sequences of sequences. By doing so, we extend the concept of sequences to study patterns in the real world: there are sequences of notes in music, sequences of operations in computing, sequences of events in history, etc. Can you think of a sequence in which appears in your day-to-day life?

Let $\{t_n\} = 1, 3, 5, 7, ...$ be the sequence of odd whole numbers. We can describe this sequence in different ways:

- Words: the sequence of odd whole numbers
- Listing out the terms: $t_1 = 1$, $t_2 = 3$, etc.
- **Closed-form formula**: $t_n = 2n - 1$ (also called closed formula or closed form)
- **Recursive formula**: $t_n = t_{n-1} + 2$

The difference between a closed-form formula and a recursive formula is that, when using the closed form, we are able to calculate the n^{th} term directly (the only variable is n). On the other hand, a formula is called recursive if it uses previous terms in the sequence (eg. if t_n is calculated using t_{n-1} or t_{n-2}).

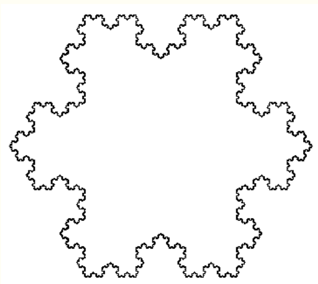
Recursive Sequences

Introduction to Recursion

Recursion is the concept of defining something in terms of itself. For example, a dictionary provides a recursive definition of the English language since every word is defined using other English words.

Fractals - Example of Recursion in Mathematics

Fractals are complex recursive geometric shapes which have a repeating pattern when 'zooming in'.



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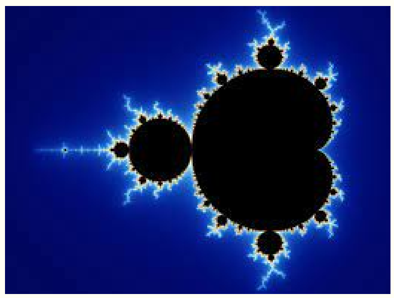

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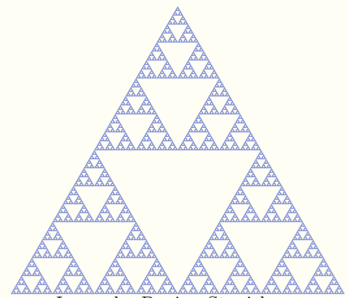

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For example, here is a video showing what happens when you zoom into the Mandelbrot set. These types of patterns occur throughout nature:

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When we think about patterns, eg. 1, 2, 4, 8, 16, ... from the warm-up, we often consider how to obtain a number using previous numbers. That is, we often look at patterns, and thus sequences, from a recursive perspective.

A **recursive sequence** is a sequence in which each term can be defined using previous terms in the sequence. The formula relating a term to the previous terms is called a **recursive formula**.

For example, here are some recursive formulas:

- $a_{n+1} = a_n$ (for $n \geq 1$)
- $b_n = b_{n-1} + 967$ (for $n \geq 2$)
- $c_{n+1} = n \times c_n$ (for $n \geq 94$)
- $d_{n+1} = d_1 + d_{n-1} - d_{n-6}$ (for $n \geq 7$)

The same sequence can have different recursive formulas. If $\{t_n\} = 1, 2, 3, 4, \ldots$ is the sequence of positive integers then $t_n = t_{n-1} + 1$ and $t_n = t_{n-2} + 2$ are both valid recursive formulas.

In order to properly define a recursive sequence, we need to know at least one of the terms. Typically we do this by providing the first term in the sequence. We also need ensure that the sequence we are defining makes sense. Formally, we say a sequence is **well-defined** if every term is uniquely defined.

Example 1

Determine whether the following sequences are well-defined:

i) $\{F_n\}$ defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

ii) $\{a_n\}$ defined by $a_1 = 1$ and $a_{2n} = 2a_n$ for $n \geq 1$.

iii) $\{a_n\}$ defined by $a_1 = 1$, $a_{2n} = 2a_n$ for $n \geq 1$, $a_n = a_{n-2} + 3$ for $n \geq 3$.

iv) $\{x_n\}$ defined by $x_{2n-1} = 2n - 1$ for $n \geq 1$ and $x_{2n} = 2x_n$ for $n \geq 1$.

Example 1 Solution

i) No. F_1 is not defined.

ii) No. a_3 is not defined.

iii) No. $a_4 = a_2 + 3 = 2a_1 + 3 = 2 \times 1 + 3 = 5$ and $a_4 = 2a_2 = 2 \times 2a_1 = 4 \times 1 = 4$ but $4 \neq 5$.

iv) Yes. If n is odd, we can use the first formula to find $x_n = n$. If n is even, use the second formula to find x_n (using $x_{\frac{n}{2}}$).

Closed-form Formula

Finding a closed-form formula when we only know the recursive definition of a sequence is very tricky. When faced with this type of problem, the first step is to write out the start of the sequence and try to guess a formula. If that doesn't work, we can try to 'unroll' the recursive formula to get more insights about the sequence.

Example 2

Find a closed-form formula for $\{t_n\}$ where $t_1 = 9$ and $t_{n+1} = t_n + 2$ for $n \geq 1$.

Example 2 Solution

Let's look at the first few terms:

$$9, 11, 13, 15, 17, ...$$

We are adding 2 to each to each term to get the next, but how can we guess the closed form? Let's work backwards using the recursive formula:

$$\begin{aligned} t_n &= t_{n-1} + 2 \\ &= (t_{n-2} + 2) + 2 \\ &= (((t_{n-3} + 2) + 2) + 2) \\ &\vdots \\ &= ((((((t_1 + 2) + 2) + ...) + 2) + 2) + 2) \end{aligned}$$

How many times will we be adding 2 to t_1?

n	2	3	4	5	6	7
# of times we add 2	1	2	3	4	5	6

We can now reasonably guess that

$$t_n = t_1 + 2(n-1) = 9 + 2(n-1) = 7 + 2n$$

Our final answer is $t_n = 2n + 7$. Note we can verify this answer by checking that it satisfies the original definition of the sequence:

$$t_1 = 7 + 2(1) = 9$$

$$t_n = 7 + 2n = 7 + 2(n-1) + 2 = t_{n-1} + 2$$

Arithmetic Sequences

An **arithmetic sequence** is a sequence where the difference between any two (consecutive) terms is constant. For example, assuming the following patterns continue,

- 2, 5, 8, 11, 14, ... we say that the **common difference** is 3
- 4, 4, 4, 4, 4, ... has common difference 0
- 9, 7, 5, 3, 1, ... has common difference -2

Suppose $\{t_n\}$ is an arithmetic sequence and let d be its common difference. Then $\{t_n\}$ satisfies the recursive formula

$$t_n = t_{n-1} + d$$

(for $n \geq 2$). A starting term t_1 and a common difference d uniquely define an arithmetic sequence:

$$t_1,\ t_1 + d,\ t_1 + 2d,\ t_1 + 3d,\ t_1 + 4d,\ t_1 + 5d,\ t_1 + 6d,\ t_1 + 7d,\ \ldots$$

We can use a technique similar to the one in Example 2 in order to find the general closed-form of an arithmetic sequence:

$$t_n = t_1 + d \times (n - 1)$$

We can check that

$$t_1 = t_1 + (1 - 1) \times d = t_1$$

$$t_n = t_1 + d \times (n - 1) = t_1 + d \times (n - 2) + d = t_{n-1} + d$$

Exercise 3

On the first day of registration for the Math Squares grade 7/8 class, 19 students register. On every following day, 9 students register.

a) How many days will it take for the class reach 40 students?

b) Let t_n be the number of students after day n, where $t_1 = 19$. Find a recursive formula for t_n.

c) Find a closed form for t_n.

d) If the class has 900 available seats, how many days will it take to fill up?

Exercise 3 Solution

a) Using the following table,

n	# of students after day n
1	19
2	28
3	37
4	46

we see that it will take 4 days for the class to reach 40 students.

b) Since 9 students register every day (after the first), $t_{n+1} = t_n + 9$ for $n \geq 1$.

c) From part b, $\{t_n\}$ is an arithmetic sequence. Therefore, a closed-form formula for t_n is

$$t_n = t_1 + d(n-1) = 19 + 9(n-1) = 10 + 9n$$

d) We want to find the first t_n such that $t_n \geq 900$. Using part c, this is the same as finding the first n such that

$$10 + 9n \geq 900$$

We can add and subtract from both sides of an inequality in the same way we can do so with an equation. Rearranging, the above inequality is equivalent to

$$9n \geq 890$$

Notice $9 \times 100 = 900$ so $9 \times 99 = 900 - 9 = 891$ and $9 \times 98 = 891 - 9 = 882$. Therefore, the smallest n such that $t_n \geq 900$ is $n = 99$ and so the class will fill up on the 99^{th} day.

Geometric Sequences

A **geometric sequence** is a sequence where the ratio between consecutive terms is constant. For example,

- 2, 4, 8, 16, 32, ... we say that the **common ratio** is 2
- 4, 4, 4, 4, 4, ... has common ratio 1
- 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, ... has common ratio $\frac{1}{3}$

Suppose $\{t_n\}$ is a geometric sequence with common ratio r. We obtain the recursive formula

$$t_{n+1} = rt_n$$

A geometric sequence is characterized by its first term t_1 and common ratio r. Given these two values, the first terms of the corresponding geometric sequence are

$$t_1, \quad t_1r, \quad t_1r^2, \quad t_1r^3, \quad t_1r^4, \quad t_1r^4, \quad t_1r^5, \ldots$$

Recall: r^n is $r \times r \times \ldots \times r$ where r is multiplied by itself n times.

Once again, let's find a closed form for this geometric sequence. According to the pattern shown in the first 6 terms, the closed-form formula for a geometric sequence is

$$t_n = t_1r^n$$

This is consistent with the recursive formula for a geometric sequence since

$$t_1 = t_1r^0 = t_1$$
$$t_n = rt_{n-1} = r \times t_1r^{n-1} = t_1r^n$$

Exercise 4

a) Let $\{a_n\}$ be a geometric sequence with $a_2 = 20$ and $a_4 = 125$. If $a_1 > 0$, find a_1.

b) Let $\{b_n\}$ be a geometric sequence. If $b_1 = b_3 = 9$, find all possible values of b_{2022}.

c) Let $\{c_n\}$ be a geometric sequence with common ratio 99. If $c_2 = 99$, find a closed formula for c_n.

d) Let $\{d_n\}$ be a geometric sequence and let a be a nonzero whole number. Is $\{d_{an}\}$ geometric ($an = a \times n$)? If so find its common ratio, otherwise find a counterexample.

Exercise 4 Solution

a) Let r be the common ratio. We have

$$\frac{25}{4} = \frac{125}{20} = \frac{a_4}{a_2} = \frac{a_1 r^3}{a_1 r} = r \times r$$

Since $a_1 > 0$ and $a_2 > 0$, we must have $r > 0$. Therefore, $r = \frac{5}{2}$. We can conclude

$$a_1 = \frac{a_2}{r} = 20 \times \frac{2}{5} = 8$$

b) Let r be the common ratio. Since $b_3 = b_1$, $r^2 = 1$ so $r = \pm 1$. If $r = 1$ then every term in the sequence is equal to 9. If $r = -1$ then every odd term in the sequence is equal to 9 and every even term is equal to -9. We can conclude that b_{2022} could be either 9 or -9.

c) $c_1 = \frac{c_2}{99} = 1$. Therefore, a closed formula for c_n is

$$c_n = 99^{n-1}$$

d) Let r be the common ratio of $\{d_n\}$. For any $n \geq 1$, we have

$$\frac{d_{a(n+1)}}{d_{an}} = \frac{d_1 r^{a(n+1)}}{d_1 r^{an}} = \frac{r^{an} \times r^a}{r^{an}} = r^a$$

therefore $\{d_{an}\}$ is geometric with common ratio r^a.

There are plenty of other sequences (or family of sequences) throughout mathematics. Check out the Online Encyclopedia of Integer Sequences, an encyclopedia of all integer sequences which mathematicians consider interesting (an integer is either a negative whole number, 0, or a positive whole number).

Fibonacci Sequence

Perhaps the most famous example of a recursive sequence is the Fibonacci sequence:

The Fibonacci sequence $\{F_n\} = F_1, F_2, ...$ is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It starts with

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ ...$$

The oldest record of this sequence is in Indian literature around 200 B.C. It is named after the Italian mathematician Fibonacci, who introduced the sequence to Western society through a problem about rabbit population growth. However, this sequence only became popular in the 19th century when mathematicians started figuring out its mathematical properties. The sequence is now commonly used as an example of how mathematical patterns appear in nature (more on that below).

The closed form of the Fibonacci sequence is:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Try calculating F_3 using this formula. Due to the complexity of the above formula, we typically prefer to work with the recursive formula instead.

The Fibonacci numbers have a number of 'nice' identities and can be used to illustrate recursive concepts in combinatorics and number theory. For example, every natural number can be uniquely written as a sum of non-consecutive natural numbers.

Exercise 5

Calculate $1 + F_1 + ... + F_n$ for $n = 1, 2, 3, 4, 5$. Do you notice a pattern?

Exercise 5 Solution

n	1	2	3	4	5
$1 + F_1 + ... + F_n$	2	3	5	8	13

Each of these numbers is a Fibonacci number! That is, assuming the pattern continues,

$$1 + F_1 + ... + F_n = F_{n+2} \text{ for } n \geq 1$$

We can check that this identity makes sense:

$$1 + F_1 + ... + F_n + F_{n+1} = (1 + F_1 + ... + F_n) + F_{n+1} = F_{n+2} + F_{n+1} = F_{n+3}.$$

The Golden Ratio

Consider the table

n	F_n	F_{n+1}	$\frac{F_{n+1}}{F_n}$
1	1	1	1
2	1	2	2
3	2	3	1.5
4	3	5	1.6666...
5	5	8	1.6
6	8	13	1.625
7	13	21	1.615384
8	21	34	1.6190476

The last column is getting closer and closer (converging) to a number called the golden ratio. We use the greek letter ϕ to represent this number:

$$\phi := \frac{1 + \sqrt{5}}{2} = 1.61803398875...$$

As the Fibonacci numbers get bigger, they behave almost like a geometric sequence with common ratio ϕ. In fact, we can rewrite the closed form of the Fibonacci numbers as

$$F_n = \frac{\phi^n - (1 - \phi)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}(1 - \phi)^n$$

$(1 - \phi)^n$ approaches 0 as n gets large. Therefore, for large n, $F_n \approx \frac{1}{\sqrt{5}}\phi^n$.

ϕ itself has a few cool properties. We can verify that ϕ satisfies:

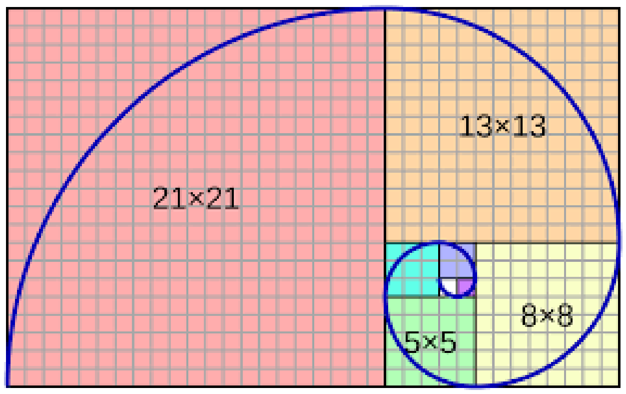
$$\phi = 1 + \frac{1}{\phi}$$

Therefore,

$$\phi = 1 + \cfrac{1}{1 + \frac{1}{\phi}} = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{\phi}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}$$

The Fibonacci Spiral & The Golden Spiral

Let's construct a spiral using the Fibonacci numbers. Start with two 1×1 squares placed beside each other. Next, continue and add $F_n \times F_n$ squares to obtain the following figure



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This can be infinitely extended to form the famous Fibonacci spiral, sometimes called the golden spiral. This spiral can be seen throughout nature, art, and architecture:

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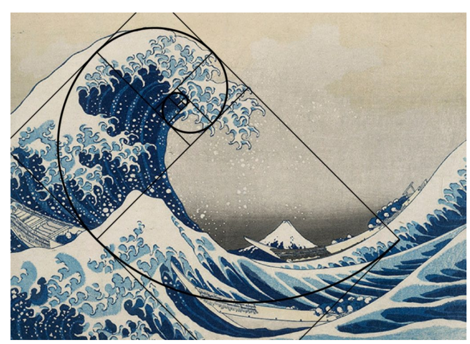

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